

A Family of Recurrence Generated Functions Based on “Half–Hyperbolic Tangent Activation Function”

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Abstract: In this note we construct a family of recurrence generated parametric half hyperbolic tangent activation functions. We prove precise upper and lower estimates for the Hausdorff approximation of the sign function by means of this family. Numerical examples, illustrating our results are given.

Keywords: Parametric Hyperbolic Tangent Activation Function (PHTA), Parametric Half Hyperbolic Tangent Activation Function (PHHTA), Sign Function, Hausdorff Distance

1. Introduction

Sigmoidal functions (also known as “activation functions”) find multiple applications to neural networks [1]–[6], [15]–[19].

We study the distance between the sign function and a special class of sigmoidal functions, so-called parametric activation functions.

The distance is measured in Hausdorff sense, which is natural in a situation when a sign function is involved. Precise upper and lower bounds for the Hausdorff distance are reported.

Since then the logistic function finds applications in many scientific fields, including biology, ecology, population dynamics, chemistry, demography, economics, geoscience, mathematical psychology, probability, sociology, political science, financial mathematics, statistics, fuzzy set theory, insurance mathematics to name a few [7]–[14], [22]–[23].

Another application area is medicine, where the logistic function is used to model the growth of tumors or to study pharmacokinetic reactions.

Constructive approximation by superposition of sigmoidal functions and the relation with neural networks and radial basis functions approximations is discussed in [19].

Any neural net element computes a linear combination of its input signals, and uses a logistic function to produce the result; often called “activation” function [20]–[21].

2. Preliminaries

The following are common examples of activation functions:
a) logistic

$$\sigma_1(t) = \frac{1}{1 + e^{-t}}; \tag{1}$$

b) Parametric Hyperbolic Tangent Activation (PHTA) function [23]

$$\sigma_2(t) = \frac{e^{\beta t} - e^{-\beta t}}{e^{\beta t} + e^{-\beta t}} = 1 - \frac{2e^{-\beta t}}{e^{\beta t} + e^{-\beta t}}, \quad t \in \mathbb{R}, \beta \geq 1; \tag{2}$$

c) Parametric Half Hyperbolic Tangent Activation (PHHTA) function [56]

$$\sigma_3(t) = \frac{1 - e^{-\beta t}}{1 + e^{-\beta t}}, \quad t \in \mathbb{R}, \beta \geq 1. \tag{3}$$

In [24] the authors create the binary logistic regression model as to find the optimal vector $\beta = [\beta_0, \beta_1, \dots, \beta_n]$ that best fits

$$y = \begin{cases} 1, & \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_n x_n + \varepsilon > 0 \\ 0, & \text{otherwise} \end{cases}$$

here ε represents the error.

Evidently, in (1) t can be regarded as a variable, which is a linear weighted combination of independent variable $x = [x_1, \dots, x_n]$ as $t \leftarrow \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_n x_n$.

Thus, the binary logistic model is [24]:

$$F(x) = \frac{1}{1 + e^{-t(\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_n x_n)}} \tag{4}$$

where $F(x)$ represents the probability of dependent variable $y = 1$.

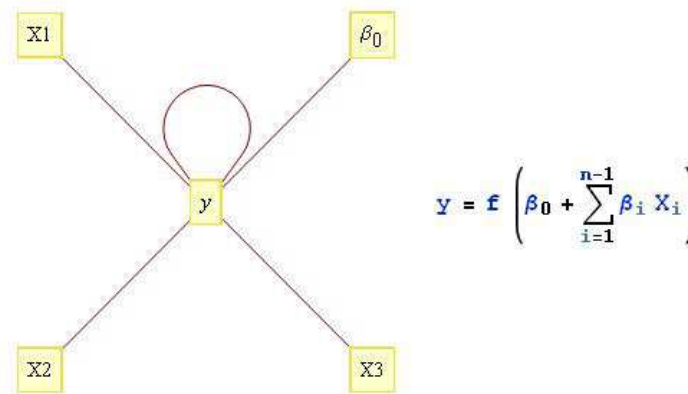


Figure 1. Nonlinear, parametrized function with restricted output range.

Training a multilayer perceptron with algorithms employing global search strategies has been an important research direction in the field of neural networks.

Multi-layer perceptrons are feed forward neural networks featuring universal approximation properties used both in regression problems.

The standard feed forward networks with only a single hidden layer can approximate any continuous function uniformly on any compact set and any measurable function to any desired degree of accuracy [25]–[28].

The nonlinear, parametrized function with restricted output range is visualized on Figure 1.

It is straightforward to extend this analysis to networks with multiple hidden layers.

- For recurrent neural networks are typical:
- a) stable outputs may be more difficult to evaluate;
- b) unexpected behavior (chaos, oscillation).

A survey of neural transfer activation functions can be found in [29].

Moreover, the nodes in the hidden layer are supposed to have a sigmoidal activation function which may be one of the following:

- a) logistic sigmoid

$$\sigma_1(net) = \frac{1}{1 + e^{-\beta net}}; \tag{5}$$

- b) hyperbolic tangent

$$\sigma_2(net) = \frac{e^{\beta net} - e^{-\beta net}}{e^{\beta net} + e^{-\beta net}} \tag{6}$$

- c) half hyperbolic tangent

$$\sigma_3(net) = \frac{1 - e^{-\beta net}}{1 + e^{-\beta net}} \tag{7}$$

where net denotes the input to a node and β is the slope parameter of the sigmoids.

Definition 1 Define the logistic (Verhulst) function v on \mathbb{R} as

$$v_0(k; t) = \frac{1}{1 + e^{-kt}}. \tag{8}$$

Note that the logistic function (8) has an inflection at its “center” $(0, 1/2)$ and its slope κ at 0 is equal to $k/4$.

Definition 2 The (basic) step function is:

$$h_0(t) = \begin{cases} 0, & \text{if } t < 0, \\ 1/2, & \text{if } t = 0, \\ 1, & \text{if } t > 0, \end{cases} \tag{9}$$

usually known as *Heaviside step function*.

Definition 3 The signum function of a real number t is defined as follows:

$$sgn(t) = \begin{cases} -1, & \text{if } t < 0, \\ 0, & \text{if } t = 0, \\ 1, & \text{if } t > 0. \end{cases} \tag{10}$$

Definition 4 [30], [31] The Hausdorff distance (the H-distance) [30] $\rho(f, g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,

$$\rho(f, g) = \max \left\{ \sup_{A \in F(f)} \inf_{B \in F(g)} \|A - B\|, \sup_{B \in F(g)} \inf_{A \in F(f)} \|A - B\| \right\}, \tag{11}$$

wherein $\|\cdot\|$ is any norm in \mathbb{R}^2 , e. g. the maximum norm $\|(t, x)\| = \max\{|t|, |x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in \mathbb{R}^2 is $\|A - B\| = \max(|t_A - t_B|, |x_A - x_B|)$.

Let us point out that the Hausdorff distance is a natural measuring criteria for the approximation of bounded discontinuous functions [8], [32].

Kyurkchiev [33] consider the following family of recurrence generated sigmoidal logistic functions

$$v_{i+1}(t) = \frac{1}{1 + k_{i+1} e^{-k(t+v_i(t))}}, \quad i = 0, 1, 2, \dots, \quad (12)$$

with

$$v_{i+1}(0) = \frac{1}{2}, \quad i = 0, 1, 2, \dots, \quad (13)$$

based on the Verhulst logistic function $v_0(t)$. From (12) we

have $k_{i+1} = e^{\frac{k}{2}}$ for $i = 0, 1, 2, \dots$.

Denote the number of recurrences by p .

The Hausdorff approximation of the Heaviside step function by family of recurrence generated sigmoidal functions of the form (12) is considered in [33] and the following is proved:

Theorem A. [33] For given p , the H-distance $d_p(k)$ between the function h_0 and the function v_p can be expressed in terms of the rate parameter k for any real $k \geq e$ as follows:

$$\begin{aligned} d_{l_p} &= \frac{1}{\frac{1}{2^{2p}} \left(k^{p+1} + \sum_{i=0}^p 2^{2(i+1)} k^{p-i} \right)} < d_p \\ &< \frac{\ln \left(\frac{1}{2^{2p}} \left(k^{p+1} + \sum_{i=0}^p 2^{2(i+1)} k^{p-i} \right) \right)}{\frac{1}{2^{2p}} \left(k^{p+1} + \sum_{i=0}^p 2^{2(i+1)} k^{p-i} \right)} = d_{r_p}. \end{aligned} \quad (14)$$

Iliev, Kyurkchiev and Markov [55] consider the following family of recurrence generated parametric activation functions

$$\gamma_{i+1}(t) = 1 - \frac{2e^{-\beta(t+\gamma_i(t))}}{e^{\beta(t+\gamma_i(t))} + e^{-\beta(t+\gamma_i(t))}}, \quad i = 0, 1, 2, \dots; \beta \geq 1, \quad (15)$$

with

$$\gamma_0(t) = 1 - \frac{2e^{-\beta t}}{e^{\beta t} + e^{-\beta t}}; \quad \gamma_0(0) = 0. \quad (16)$$

Evidently, $\gamma_{i+1}(0) = 0$ for $i = 0, 1, 2, \dots$.

The following Theorem gives upper and lower bounds for the Hausdorff approximation d of the sgn function (10) by the family (15).

Theorem B. [55] For given p , the H-distance d_p between the sgn function and the function γ_p the following inequalities hold for $\beta \geq 3$:

$$d_{l_p} = \frac{1}{1.5 \sum_{i=0}^{p+1} \beta^i} < d_p < \frac{\ln \left(1.5 \sum_{i=0}^{p+1} \beta^i \right)}{1.5 \sum_{i=0}^{p+1} \beta^i} = d_{r_p}. \quad (17)$$

3. Main Results

In this Section we construct a family of recurrence generated parametric activation functions based on $\sigma_3(t)$.

We prove precise upper and lower estimates for the Hausdorff approximation of the sign function by means of this family.

3.1. The Family of Recurrence Generated Parametric Half Hyperbolic Tangent Activation (PHHTA) Functions

We consider the following family of recurrence generated parametric activation functions:

$$\delta_{i+1}(t) = \frac{1 - e^{-\beta(t+\delta_i(t))}}{1 + e^{-\beta(t+\delta_i(t))}}, \quad i = 0, 1, 2, \dots; \beta \geq 1, \quad (18)$$

with

$$\delta_0(t) = \frac{1 - e^{-\beta t}}{1 + e^{-\beta t}}; \quad \delta_0(0) = 0. \quad (19)$$

Evidently, $\delta_{i+1}(0) = 0$ for $i = 0, 1, 2, \dots$.

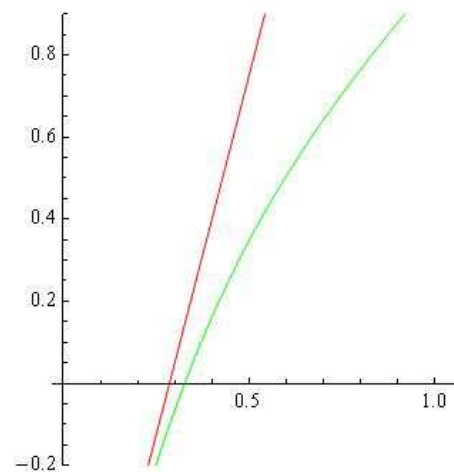


Figure 2. The functions $F_0(d_0)$ and $G_0(d_0)$ for $\beta = 5$.

Denote the number of recurrences by p .

The recurrence generated (PHHTA)-functions $\delta_0(t)$, $\delta_1(t)$, $\delta_2(t)$ and $\delta_3(t)$ for various β are visualized on Figure 3–Figure 4.

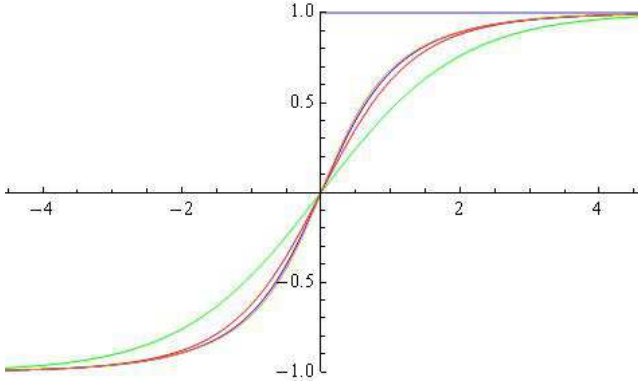


Figure 3. Approximation of the $\text{sgn}(t)$ by (PHHTA)-functions for $\beta=1$; The graphics of recurrence generated (PHHTA)-functions: δ_0 (green), δ_1 (red), δ_2 (blue) and δ_3 (orange); Hausdorff distance: $d_0=0.674832$, $d_1=0.588575$, $d_2=0.55769$, $d_3=0.545712$.

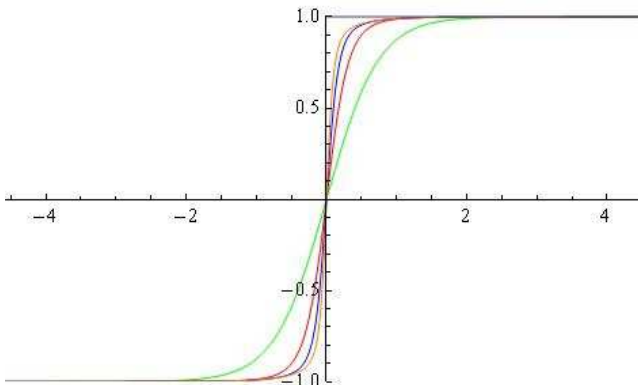


Figure 4. Approximation of the $\text{sgn}(t)$ by (PHHTA)-functions for $\beta=2.71828$; The graphics of recurrence generated (PHHTA)-functions: δ_0 (green), δ_1 (red), δ_2 (blue) and δ_3 (orange); Hausdorff distance: $d_0=0.45295$, $d_1=0.287317$, $d_2=0.21027$, $d_3=0.168516$.

3.2. Approximation Issues

We study the Hausdorff distance d between the sign function and the family of (PHHTA)-functions (18).

Special case. Let $p=0$.

The H -distance $d_0(\text{sgn}(t), \delta_0(t))$ between the sgn function and the function δ_0 satisfies the relation:

$$\delta_0(d_0) = \frac{1 - e^{-\beta d_0}}{1 + e^{-\beta d_0}} = 1 - d_0. \tag{20}$$

The following Theorem gives upper and lower bounds for d_0

Theorem 3.1. For the Hausdorff distance d_0 between the sgn function and the function δ_0 the following inequalities hold for $\beta \geq 5$:

$$d_{l_0} = \frac{1}{\frac{1}{2}(2 + \beta)} < d_0 < \frac{\ln\left(\frac{1}{2}(2 + \beta)\right)}{\frac{1}{2}(2 + \beta)} = d_{r_0}. \tag{21}$$

Proof. We define the functions

$$F_0(d_0) = \frac{1 - e^{-\beta d_0}}{1 + e^{-\beta d_0}} - 1 + d_0 \tag{22}$$

$$G_0(d_0) = -1 + \frac{1}{2}(2 + \beta)d_0 \tag{23}$$

From Taylor expansion

$$F_0(d_0) - G_0(d_0) = O(d_0^2)$$

we see that $G_0(d_0)$ approximates $F_0(d_0)$ with $d_0 \rightarrow 0$ as $O(d_0^2)$ (cf. Figure 2).

In addition $G'_0(d_0) > 0$ and for $\beta \geq 5$

$$G_0(d_{l_0}) = 0; \quad G_0(d_{r_0}) > 0.$$

This completes the proof of the inequalities (21).

General Case.

Theorem 3.2 For given p , the H -distance d_p between the sgn function and the function δ_p the following inequalities hold for $\beta \geq 5$:

$$d_{l_p} = \frac{1}{\frac{1}{2^{p+1}} \left(\beta^{p+1} + \sum_{i=0}^p 2^{i+1} \beta^{p-i} \right)} < d_p < \frac{\ln\left(\frac{1}{2^{p+1}} \left(\beta^{p+1} + \sum_{i=0}^p 2^{i+1} \beta^{p-i} \right)\right)}{\frac{1}{2^{p+1}} \left(\beta^{p+1} + \sum_{i=0}^p 2^{i+1} \beta^{p-i} \right)} = d_{r_p}. \tag{24}$$

Proof. We note that the function

$$G_p(d_p) = -1 + \frac{1}{2^{p+1}} \left(\beta^{p+1} + \sum_{i=0}^p 2^{i+1} \beta^{p-i} \right) d_p$$

approximates $F_p(d_p)$ with $d_p \rightarrow 0$ as $O(d_p^2)$.

In addition $G'_p(d_p) > 0$ and for $\beta \geq 5$

$$G_p(d_{l_p}) = 0; G_p(d_{r_p}) > 0.$$

This completes the proof of the inequalities (24).

4. An Analysis of Recurrence Generated Sigmoidal Functions Based on the Richards Type Function

In [57] Turner, Blumenstein and Sebaugh introduced the generalization of the logistic law growth of the form:

$$x' = \beta x \left(\frac{\tau(k) - \tau(x)}{\tau(k)} \right). \tag{25}$$

Here $\tau(\cdot)$ denotes the process of operating on any argument (\cdot) with the operator τ , β is proportionally constant, or "intrinsic growth coefficient".

Let $\tau(x) = x$, then equation (25) leads to the ordinary logistic law of growth

$$x' = \beta x \frac{k - x}{k} = \frac{\beta}{k} x(k - x). \tag{26}$$

Let $\tau(\cdot)$ is the power function $(\cdot)^m$, where $m > 0$.

Thus (25) becomes

$$x' = \beta x \frac{k^m - x^m}{k^m} = \beta k^{-m} (k^m - x^m). \tag{27}$$

with the solution [58]:

$$x = \frac{k}{\left(1 + (k^m x_0^{-m} - 1)e^{-\beta m t} \right)^{\frac{1}{m}}}. \tag{28}$$

An alternative form of a generalized logistic equation has been given by Nelder in [59]:

$$x(t) = \frac{1}{\left(1 + e^{-\frac{kt + \lambda}{\theta}} \right)^{\theta}}, \theta > 0. \tag{29}$$

For other results, see [60].

Garcia [61], [62] presented a generalized model, depending on two shape parameters, that includes most of the common growth functions as special cases.

The general equation is

$$y = B^{-1} \left[B^{-1}(t, b), a \right], \tag{30}$$

where B is the negative Box-Cox transformation [63]

$$B(x, c) = \frac{1 - x^c}{c}, c \neq 0$$

and $B(x, 0) = -\ln x$. Or $B^{-1}(x, c) = (1 - cx)^{\frac{1}{c}}, c \neq 0$, and $B^{-1}(x, 0) = e^{-x}$.

Special case:

$$y = \frac{1}{(1 + Ae^{-t})^\alpha}, \alpha > 0 \text{ (Richards-type)} \tag{31}$$

Let us consider the following family of recurrence generated sigmoidal logistic functions (cf. Figure 5)

$$\omega_i(t) = \frac{1}{\left(1 + k_i e^{-(t + \omega_{i-1}(t))} \right)^\alpha}, i = 1, 2, \dots, \tag{32}$$

with

$$\omega_i(0) = \frac{1}{2^\alpha}, i = 1, 2, \dots, \tag{33}$$

based on the Richards-type function $\omega_0(t)$

$$\omega_0(t) = \frac{1}{(1 + e^{-t})^\alpha}$$

with

$$\omega_0(0) = \frac{1}{2^\alpha}.$$

From (33) we have $k_i = e^{\frac{1}{2^\alpha}}$ for $i = 1, 2, \dots$.

Definition. The associate to the family $\omega_i(t)$ cut function $C_{\omega_i(t)}$ is defined by

$$C_{\omega_i(t)}(t) = \begin{cases} 0, & \text{if } t < t_1^i, \\ \omega_i'(t_i^*)(t - t_i^*) + \omega_i(t_i^*), & \text{if } t_1^i \leq t < t_2^i, \\ 1, & \text{if } t \geq t_2^i. \end{cases} \tag{34}$$

$\omega_i(t)$ has an inflection at point $T_i^*(t_i^*, \omega_i(t_i^*))$.

The straight line $y_i = \omega_i'(t_i^*)(t - t_i^*) + \omega_i(t_i^*)$ cross the lines $y = 0$ and $y = 1$ at the points t_1^i and t_2^i respectively.

We next focus on the approximation of the cut function $C_{\omega_i(t)}$ by $\omega_i(t)$.

Note that the slope of the function $C_{\omega_i(t)}$ on the interval $\Delta_i = [t_1^i, t_2^i]$ is $\omega_i'(t_i^*)$.

Then, noticing that the largest uniform distance ρ_i between the cut and $\omega_i(t)$ functions is achieved at the endpoints of the underlying interval Δ_i we have the following

Theorem. The function defined by (32):
 i) is the function of best uniform approximation to function $C_{\omega_i(t)}$ in the interval Δ_i ;
 ii) approximates the cut function $C_{\omega_i(t)}$ in uniform metric with an error

$$\rho_i = \max \{ \omega_i(t_1^i), 1 - \omega_i(t_2^i) \}. \tag{35}$$

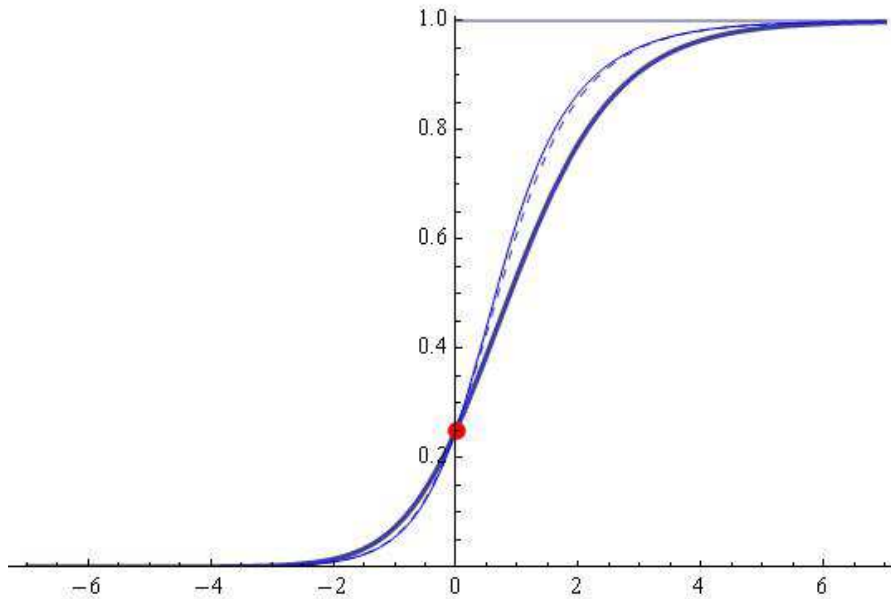


Figure 5. The parameter $\alpha = 2$. The graphics: function - ω_0 (thick), recurrence generated function - ω_1 (dashed) and recurrence generated function - ω_2 (blue).

5. Conclusion

In biologically plausible neural networks, the activation functions represent the rate of action potential firing in the cell [35].

A family of recurrence generated parametric activation functions is introduced finding application in neural network theory and practice.

Theoretical and numerical results on the approximation in Hausdorff sense of the sgn function by means of functions belonging to the family are reported in the paper.

We propose a software module within the programming environment *CAS Mathematica* for the analysis of the considered family of recurrence generated (PHHTA) functions.

The module offers the following possibilities:

- generation of the activation functions under user defined values of the parameter β and number of recursions p ;
- calculation of the H-distance d_p , $p = 0, 1, 2, \dots$, between the sgn function and the activation functions

$$\delta_0, \delta_1, \delta_2, \dots, \delta_p;$$

- software tools for animation and visualization.

The Hausdorff approximation of the interval step function by the logistic and other sigmoidal functions is discussed from various approximation, computational and modelling aspects in [36]–[54].

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